Solving Integral Curves and its Applications

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Abstract

In this paper we study methods for finding integral curves and surfaces of vector fields. And then we express the applications to plasma physics and to solenoidal vector fields.

Integral Curves of Vector Fields

Let $\mathbf{V}(x,y,z)=(\mathbf{P}(x,y,z),\mathbf{Q}(x,y,z),\mathbf{R}(x,y,z))$ be a vector field defined in some domain of R^3. We will deal only with the vector fields **V** in domains Ω in which the following two conditions are satisfied:

(a) **V** is nonvanishing in Ω; i. e., the component functions **P**, **Q**, **R** of **V** do not vanish simultaneously at any point of Ω,

(b) $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in C^1(\Omega)$.

Definition

A curve C in Ω is an integral curve of the vector field **V** if **V** is tangent to C at each of its points.

With the vector field $\mathbf{V}=(P,Q,R)$ we associate the system of ordinary differential equations,

$$\frac{dx}{dt}=\mathbf{P}(x,y,z), \qquad \frac{dy}{dt}=\mathbf{Q}(x,y,z), \qquad \frac{dz}{dt}=\mathbf{R}(x,y,z). \qquad (1)$$

A solution $(x(t),y(t),z(t))$ of (1), defined for t in some interval I, may be regarded as a curve in Ω. We will call this curve a solution curve of the system (1). Obviously, every solution curve of the system (1) is an integral curve of the vector field **V**. Conversely, it can be shown, that if C is an integral curve of **V**, then there is a parametric representation

$$x=x(t),\ y=y(t), \qquad z=z(t); \qquad t\in I,$$

of C, such that $(x(t),y(t),z(t))$ is a solution of the system of equations (1). Thus, every integral curve of **V**, if parameterized appropriately, is a solution curve of the associated system of equations (1).

The integral curves of simple vector fields, such as those given by $\mathbf{V}=(1,0,0)$ and $\mathbf{V}=(x,y,z)$, can sometimes be found by geometrical intuition. However, for more complicated vector fields this is not always possible. In any case, the integral curves of a vector

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field $\mathbf{V} = (\mathbf{P}, \mathbf{Q}, \mathbf{R})$ can be found by considering these curves as solution curves of the associated system of equations (1) and by solving this system.

Since the right hand sides of the system (1) do not depend on t, it is possible to eliminate t completely and consider any two of the variables x, y, z as functions of the third. If, for example, $\mathbf{P} \neq 0$, then y and z may be considered as functions of the independent variable x, and the system (1) may be written in form

$$\frac{dy}{dx} = \frac{Q}{P}, \qquad \frac{dz}{dx} = \frac{R}{P}. \tag{2}$$

Similarly, if $\mathbf{Q} \neq 0$, or $\mathbf{R} \neq 0$, the system (1) may be written in the form

$$\frac{dx}{dy} = \frac{\mathbf{P}}{\mathbf{Q}}, \qquad \frac{dz}{dy} = \frac{\mathbf{R}}{\mathbf{Q}} \tag{3}$$

or

$$\frac{dx}{dz} = \frac{\mathbf{P}}{\mathbf{R}}, \qquad \frac{dy}{dz} = \frac{\mathbf{Q}}{\mathbf{R}}, \tag{4}$$

respectively. In order to avoid distinguishing between dependent and independent variables, it is customary to write the equivalent systems (2) - (4) in the form

$$\frac{dx}{\mathbf{P}} = \frac{dy}{\mathbf{Q}} = \frac{dz}{\mathbf{R}}. \tag{5}$$

Definition

Two functions u_1 and u_2 in $C^1(\Omega)$ which satisfy condition
$\text{grad } u_1(x, y, z) \times \text{grad } u_2(x, y, z) \neq 0, \ (x, y, z) \in \Omega,$ will be called functionally independent in Ω.

Definition

A function u in $C^1(\Omega)$ is called a first integral of the vector field $\mathbf{V} = (\mathbf{P}, \mathbf{Q}, \mathbf{R})$ (or of its associated system dx/**P** = dy/**Q** = dz/**R**) in Ω, if at each point of Ω, **V** is orthogonal to grad u, i.e.,

$$\mathbf{P}\frac{\partial u}{\partial x} + \mathbf{Q}\frac{\partial u}{\partial y} + \mathbf{R}\frac{\partial u}{\partial z} = 0 \quad \text{in } \Omega. \tag{6}$$

Equation (6) is a partial differential equation in unknown function u of three independent variables x, y, z. According to Definition 1.3, any solution of the p.d.e. (6) is a first integral of **V**.

Some Examples

(i) Let $\mathbf{V} = (1,0,0)$ be the vector field and let $\Omega = R^3$. A first integral of **V** is a solution of the equation

$$u_x = 0 . \qquad (7)$$

Any function of y and z only is a solution of this equation. For example,

$$u_1 = y , \qquad u_2 = z$$

are two solutions which are obviously functionally independent. The integral curves of **V** are described by the equations

$$y = c_1 , \qquad z = c_2 , \qquad (8)$$

and are straight lines parallel to the x-axis (see Fig. 1). The functions $y - z$ and e^{y+z} are also functionally independent first integrals, and the integral curves of **V** are also described by the equations

$$y - z = c_1 \qquad e^{y+z} = c_2 . \qquad (9)$$

Of course, different values of c_1 and c_2 must be used in (7) and (8) in order to get the same integral curve of **V**.

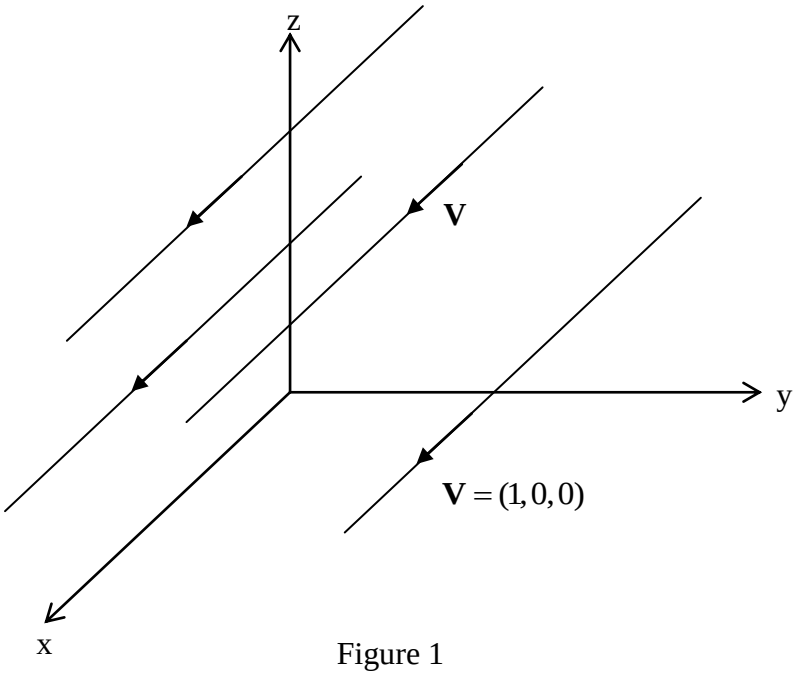

Figure 1

(ii) Let **V** be the vector field $\mathbf{V} = (x, y, z)$ and let Ω be the octant $x > 0,\ y > 0,\ z > 0$. A first integral of **V** is a solution of the equation

$$xu_x + yu_y + zu_z = 0. \qquad (10)$$

It can be found that the functions

$$u_1(x, y, z) = \frac{y}{x}, \qquad u_2(x, y, z) = \frac{z}{x}$$

are first integrals of V in Ω. Moreover, they are functionally independent in Ω since they satisfy $\text{grad } u_1(x, y, z) \times \text{grad } u_2(x, y, z) \neq 0$. Therefore, the integral curves of **V** in Ω are described by the equations

$$\frac{y}{x} = c_1, \qquad \frac{z}{x} = c_2. \qquad (11)$$

They are rays emanating from the origin (see Fig. 2) and a parametric representation of them is

$$x = t, \qquad y = c_1 t, \qquad z = c_2 t; \qquad t > 0.$$

It is easy to check by direct computation that any function of u_1 and (or) u_2 is also a first integral of **V**.

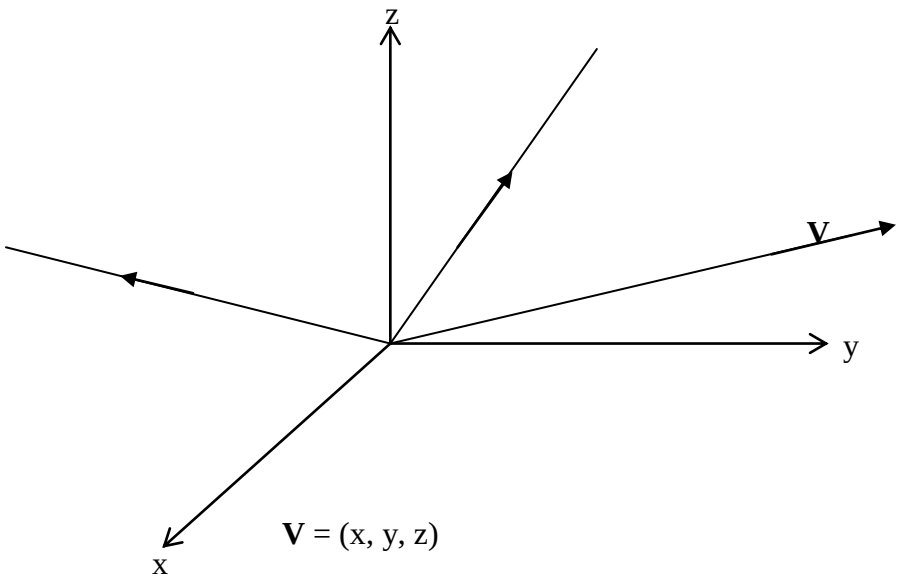

Figure 2

(iii) Let $\mathbf{V} = (y, -x, 0)$ be the vector field and let Ω be R^3 minus the z-axis. A first integral of **V** is a solution of the equation

$$yu_x - xu_y = 0.$$

It can be found that the functions.

$$u_1(x, y, z) = x^2 + y^2, \qquad u_2(x, y, z) = z$$

are two functionally independent first integrals of **V**. Therefore, the integral curves of **V** in Ω are given by

$$x^2 + y^2 = c_1, \qquad z = c_2. \tag{12}$$

Equations (12) describe circles parallel to the (x, y)-plane and centered on the z-axis.

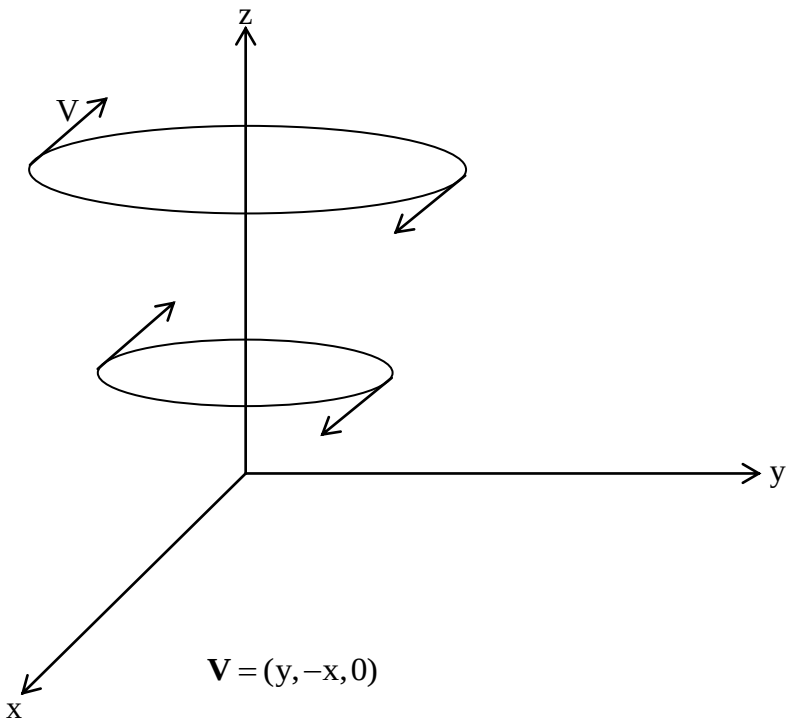

Figure 3

Definition

A surface S in a domain Ω of R^3 is an integral surface of the vector **V** if S is a level surface of a first integral of **V**; i.e., S is described by an equation of the form

$$u(x, y, z) = c \tag{13}$$

where u is a solution of the equation

$$\mathbf{P}u_x + \mathbf{Q}u_y + \mathbf{R}u_z = 0 \tag{14}$$

in Ω such that $\operatorname{grad} u \neq 0$ in Ω.

Example

We consider the vector field $V=(1,0,0)$. The corresponding equation (14) is

$$u_x = 0 \tag{15}$$

and the associated system of equations is

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dz}{0}. \tag{16}$$

The integral curves of V are given by

$$y=c_1,\ \ z=c_2$$

and they are lines parallel to the x-axis. Suppose first that the curve C lies on the $x=0$ plane and is given by the equations

$$f(y,z)=0, \qquad x=0. \tag{17}$$

Then the cylindrical surface S given by

$$f(y,z)=0 \tag{18}$$

is the integral surface of **V** containing the curve C. Next, suppose that C is an integral curve of **V** given by

$$y=y_0, \qquad z=z_0. \tag{19}$$

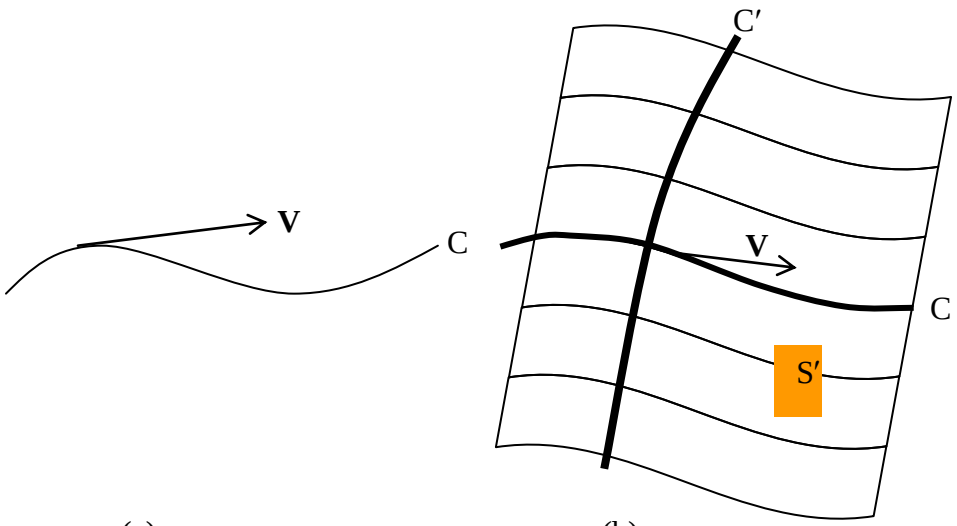

Figure 4

Let C' be any curve on the $x=0$ plane passing through the point $(0,y_0,z_0)$. Then C' is given by equations of the form (17) with the condition $f(y_0,z_0)=0$. It can be seen that the surface S' given by equation (18) is an integral surface of V containing the curve (19). In fact, any surface given by an equation of the form (18), with the function f subject only to the condition $f(y_0,z_0)=0$, is an integral surface of **V** containing the curve (19).

Application to Plasma Physics and to Solenoidal Vector Fields

Application to Plasma Physics

The basic equation of plasma physics is known as the Boltzmann equation which is used in the study of a problem known as a static boundary-layer problem,

$$mv_1 \frac{\partial f}{\partial x} + e\left(\frac{v_2}{c}\frac{d\eta}{dx} - \frac{d\varphi}{dx}\right)\frac{\partial f}{\partial v_1} - e\,\frac{v_1}{c}\,\frac{d\eta}{dx}\,\frac{\partial f}{\partial v_2} = 0. \qquad (21)$$

In equation (21) f is the unknown function of the three independent variables x, v_1 and v_2. The functions ϕ and η are given functions of the variable x only, while m, e and c are constants. The partial differential equation (21) is an equation of the form

$$\frac{dx}{mv_1} = \frac{dv_1}{e\left(\frac{v_2}{c}\frac{d\eta}{dx} - \frac{d\phi}{dx}\right)} = \frac{dv_2}{-e\frac{v_1}{c}\frac{d\eta}{dx}}. \qquad (22)$$

The equation of the first and third ratios (after canceling v_1) is an o.d.e. in x and v_2 which yields the first integral

$$f_1 = mv_2 + \frac{e}{c}\eta(x).$$

Multiplying the numerators and denominators of the second ratio in (22) by $2v_1$ and of the third ratio by $2v_2$ and adding the numerators and denominators of the resulting ratios yields the ratio

$$\frac{d(v_1^2 + v_2^2)}{-2ev_1\frac{d\phi}{dx}} \qquad (23)$$

which is also equal to the ratios (22). The equality of the ratio (23) with the first ratio in (22) (after canceling v_1) is an o.d.e. in the variables x and $\left(v_1^2 + v_2^2\right)$ which yields the first integral

$$f_2 = \frac{1}{2}m\left(v_1^2 + v_2^2\right) + e\phi(x). \qquad (24)$$

Obviously, f_1 and f_2 are functionally independent and, the general solution of (21) is given by

$$f(x_1, v_1, v_2) = \mathbf{F}\left(mv_2 + \frac{e}{c}\eta(x), \frac{1}{2}m\left(v_1^2 + v_2^2\right) + e\phi(x)\right), \qquad (25)$$

where $\mathbf{F}(f_1, f_2)$ is an arbitrary function of two variables. The first integrals f_1 and f_2 have physical meaning; f_2 is the energy of a particle of mass m and f_1 is its canonical momentum. Moreover, the pair of equations

$$f_1 = c_1, f_2 = c_2$$

determine the trajectory of the particle.

Solenoid Vector Fields

Let **V**= (**P**, **Q**, **R**)be a vector field defined in some domain Ω in R^3,with **P**, **Q**, **R**, belonging to $C^1(\Omega)$.The divergence of **V**. written div **V**, is the function defined in Ω by

$$\operatorname{div}\mathbf{V}=\frac{\partial \mathbf{P}}{\partial \mathrm{x}}+\frac{\partial \mathbf{Q}}{\partial \mathrm{y}}+\frac{\partial \mathbf{R}}{\partial \mathrm{z}}.$$

V is said to be solenoidal in Ω if $\operatorname{div}\mathbf{V}=0$ in Ω.

The curl of **V**, written curl **V**, is the vector field defined in Ω by

$$\operatorname{curl}\mathbf{V}=\left(\frac{\partial \mathbf{R}}{\partial \mathrm{y}}-\frac{\partial \mathbf{Q}}{\partial \mathrm{z}},\frac{\partial \mathbf{P}}{\partial \mathrm{z}}-\frac{\partial \mathbf{R}}{\partial \mathrm{x}},\frac{\partial \mathbf{Q}}{\partial \mathrm{x}}-\frac{\partial \mathbf{P}}{\partial \mathrm{y}}\right).$$

The following theorem finds frequent application in many areas of engineering an physics.

Theorem

Let $\mathbf{V}=(\mathbf{P},\mathbf{Q},\mathbf{R})$ be a nonvanishing vector field defined in a domain Ω of R^3, with $\mathbf{P},\mathbf{Q},\mathbf{R}$ in $C^1(\Omega)$. If **V** is solenoidal in Ω, then given any point (x_0,y_0,z_0) in Ω, there is a neighborhood Ω_0 of (x_0,y_0,z_0) and a vector field **W** with C^1 components defined in Ω_0 such that

$$\mathbf{V}(\mathrm{x},\mathrm{y},\mathrm{z})=\operatorname{curl}\ \mathbf{W}(\mathrm{x},\mathrm{y},\mathrm{z}),\quad (\mathrm{x},\mathrm{y},\mathrm{z})\in\Omega_0. \tag{26}$$

The vector field **W** is often called a vector potential for given field **V.** Before giving the proof, we will list some identities from vector calculus which will be needed in the course of the proof. Let f be a C^1 function, and let u, v be C^1 vector fields, all being defined in a common domain Ω.

Then

$$\operatorname{div}(\mathrm{f}\,\mathbf{u})=\operatorname{grad}\mathrm{f}\cdot\mathbf{u}+\mathrm{f}\ \operatorname{div}\mathbf{u}, \tag{27}$$

$$\operatorname{div}(\mathbf{u}\times\mathbf{v})=(\operatorname{curl}\mathbf{u})\cdot\mathbf{v}-(\operatorname{curl}\mathbf{v})\cdot\mathbf{u}, \tag{28}$$

$$\operatorname{curl}(\operatorname{grad}\mathrm{f})=0\ (\text{assume } \mathrm{f}\in C^2 \text{ here}), \tag{29}$$

$$\operatorname{curl}(\mathrm{f}\,\mathbf{u})=(\operatorname{grad}\mathrm{f})\times\mathbf{u}+\mathrm{f}\ \operatorname{curl}\mathbf{u}. \tag{30}$$

Proof:

Let u_1,u_2 be two first integrals of **V** which are functionally independent in some neighborhood Ω_1 of (x_0,y_0,z_0). At each point of Ω_1, the vector field **V** is parallel to $\operatorname{grad}u_1\times\operatorname{grad}u_2$, so that we can write

$$\mathbf{V}(\mathrm{x},\mathrm{y},\mathrm{z})=\lambda(\mathrm{x},\mathrm{y},\mathrm{z})(\operatorname{grad}u_1\times\operatorname{grad}u_2) \tag{31}$$

for some function λ defined in Ω_1. The function λ is C^1 since

$$\lambda = \frac{\mathbf{V}\cdot(\operatorname{grad} u_1 \times \operatorname{grad} u_2)}{\left|\operatorname{grad} u_1 \times \operatorname{grad} u_2\right|^2},$$

and u_1, u_2 are actually C^2. This smoothness of u_1 and u_2, which we have not used before, follows from the manner in which u_1 and u_2 are obtained from the system of ordinary differential equations (5). Since $\mathbf{V}$ is solenoidal in Ω_1, it follows by applying identities (27), (28), and (29) that

$$\begin{aligned} 0 &= \operatorname{div}\mathbf{V} \\ &= \operatorname{grad}\lambda\cdot(\operatorname{grad} u_1 \times \operatorname{grad} u_2) + \lambda\begin{bmatrix}(\operatorname{curl}\operatorname{grad} u_1)\cdot\operatorname{grad} u_2 \\ -(\operatorname{curl}\operatorname{grad} u_2)\cdot\operatorname{grad} u_1\end{bmatrix} \\ &= \operatorname{grad}\lambda\cdot(\operatorname{grad} u_1 \times \operatorname{grad} u_2). \end{aligned} \tag{32}$$

Equation (32) shows that $\operatorname{grad}\lambda$ is perpendicular to $\operatorname{grad} u_1 \times \operatorname{grad} u_2$ at each point of Ω_1, and so is perpendicular to $\mathbf{V}$ at each point of Ω_1, i.e.

$$\mathbf{P}\frac{\partial\lambda}{\partial x} + \mathbf{Q}\frac{\partial\lambda}{\partial y} + \mathbf{R}\frac{\partial\lambda}{\partial z} = 0 \text{ in } \Omega_1. \tag{33}$$

Thus λ is a solution of the partial differential equation and we can apply the results of that section to express λ as a function of u_1 and u_2. Explicitly, that there is a neighborhood Ω_0 of (x_0, y_0, z_0) with $\Omega_0 \subset \Omega_1$ and a C^1 function $F(u_1, u_2)$ such that

$$\lambda(x,y,z) = \mathbf{F}\left(u_1(x,y,z), u_2(x,y,z)\right), (x,y,z) \in \Omega_0. \tag{35}$$

Now, let $\mathbf{G}(u_1, u_2)$ be a function such that

$$\mathbf{F}(u_1, u_2) = \frac{\partial\mathbf{G}}{\partial u_1}(u_1, u_2). \tag{36}$$

From (31),(35) and (36) we see that in Ω_0

$$\begin{aligned} \mathbf{V} &= \left(\frac{\partial\mathbf{G}}{\partial u_1}\operatorname{grad} u_1\right)\times\operatorname{grad} u_2 \\ &= \operatorname{grad}\mathbf{G}\times\operatorname{grad} u_2. \end{aligned} \tag{37}$$

In the last line of (37) we used the identities

$$\operatorname{grad}\mathbf{G}(u_1, u_2) = \frac{\partial\mathbf{G}}{\partial u_1}\operatorname{grad} u_1 + \frac{\partial\mathbf{G}}{\partial u_2}\operatorname{grad} u_2, \quad \operatorname{grad} u_2 \times \operatorname{grad} u_2 = 0.$$

To complete the proof we need only observe that (37) can be written in the form

$$\mathbf{V} = \operatorname{curl}(\mathbf{G}\operatorname{grad} u_2)$$

because of identities (29) and (30). If we set

$$\mathbf{W} = \mathbf{G}\ \text{grad}\ u_2 \tag{38}$$

then

$$\mathbf{V} = \text{curl}\ \mathbf{W} \quad \text{in} \quad \Omega_0.$$

Example

Let $\mathbf{V} = (y, -x, 0)$ be the vector field with Ω being R^3 minus the z-axis. **V** is clearly solenoidal in Ω. Two functionally independents first integrals of **V** were found in Example (iii) to be

$$u_1 = x^2 + y^2, \qquad u_2 = z.$$

By calculating show that

$$\begin{aligned} \text{grad}\, u_1 \times \text{grad}\, u_2 &= 2(y, -x, 0) \\ &= 2\mathbf{V}, \end{aligned}$$

so that the proportionality factor λ in this case is simply

$$\lambda = \frac{1}{2}$$

The function $F(u_1, u_2)$ in (35) is

$$\mathbf{F}(u_1, u_2) \equiv \frac{1}{2},$$

and for $\mathbf{G}(u_1, u_2)$ we can take the function $(1/2)u_1$. It follows from (38) that

$$\begin{aligned} \mathbf{W} &= \frac{1}{2} u_1\, \text{grad}\, u_2 \\ &= \frac{1}{2}(x^2 + y^2)\, \text{grad}\, z \\ &= (0, 0, \frac{1}{2}(x^2 + y^2)). \end{aligned}$$

Thus, for all points of Ω we have

$$\mathbf{V} = \text{curl}(0, 0, \frac{1}{2}(x^2 + y^2)).$$

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References

[1] Zachmanoglou, E.C. and Thoe, D.W., "Introduction to Partial Differential Equations with Applications", Dover Publications, Inc., New York, (1976).

[2] Trim, D.W., "Applied Partial Differential Equations", http://16811-aiedpartial differential equations.pdf, (2013).

Coloring of Planar Graphs

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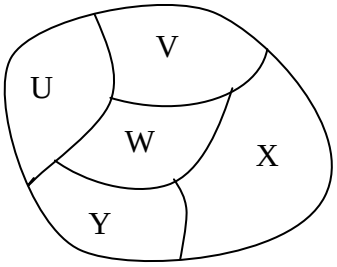
Abstract

There are many different ways to colors a graph, each with different applications. In this paper, we discuss vertex coloring, and find the least number of colors needed for a coloring of the given graph.

Introduction

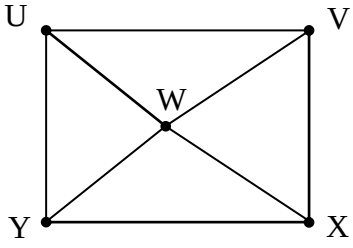
Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The resulting graph is called the **dual graph** of the map. By the way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph.

The problem of coloring the regions of the map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color. We now define the coloring of graphs.



Map

Figure 1



Dual graph of the map

Figure 2

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Terminology and Notations

A **graph** $G=(V, E)$ consists of a finite nonempty set V, called the set of vertices and a set E of unordered pair of vertices, called the set of edges. Two vertices which are incident with a common edge are **adjacent**, as are two edges which are incident with a common vertex. An edge with identical ends is called a **loop**. All linear having the same pair of end points are called **parallel edges**. A graph is **simple** if it has no loops and no parallel edges. The **degree** d(v) of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. A **walk** in G is a finite non-null sequence $W=v_0, e_1, v_1, e_2, \ldots, e_k, v_k,$ whose terms are alternately vertices and edges. A **trail** is a walk with no repeated edge. A **path** is a walk with no repeated vertex. A **cycle** is a closed trail. A graph is said to be **acyclic** if it does not contain any cycles. Let e an edge of a graph G. If $G-e$ has more components than G, then e is a **bridge** of G. Two vertices u and v of G are said to be **connected** if there is a (u, v)-path in G. A connected graph without cycles is a **tree**.

The **complete graph** on n vertices, for $n \geq 1$, which we denote K_n, is a graph with n vertices an edge joining every pair of distinct vertices. A graph $G'=(V', E')$ is a **subgraph** of a graph $G=(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Planar Graph

A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing.

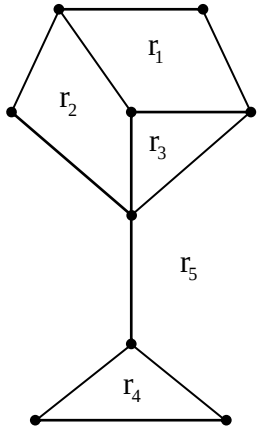

Figure 3 The region of the planar representation of a graph

A **coloring** of a simple graph is the assignment of a color to each vertex of the so that no two adjacent vertices are assigned the same color. A **chromatic number** of a graph is the least number of colors needed for a coloring of this graph. A **vertex coloring** of a graph $G=(V, E)$ is a map $c: V \rightarrow S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The element of the set S are called the available **colors**. If each colors used in one of k given colors, then we refer to the coloring as a **k-coloring**. In a k-coloring, we may then assume that it is the colors 1, 2,…, k that are being used. This k is the **chromatic number** of G; it is denoted by $\chi(G)$. A graph G with $\chi(G)=k$ is called k-chromatic; if $\chi(G) \leq k$, we call G is **k-colorable**.

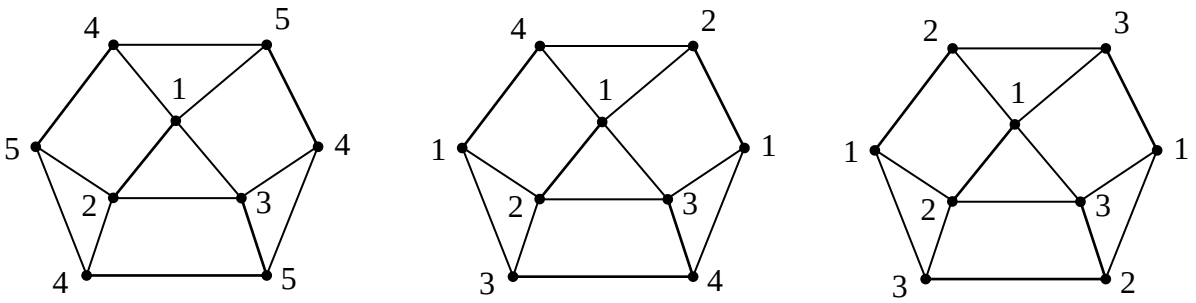

Figure 4 Coloring of a graph H

Theorem

If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.

Proof

Suppose that $\chi(G) = k$. Then there exists a k-coloring e of G. Since c assigns distinct colors to every two adjacent vertices of G, the coloring c also assigns distinct colors to every two adjacent vertices of H. Therefore, H is k-colorable and so $\chi(H) \leq k = \chi(G)$.

Theorem

If G is a graph whose largest vertex-degree is ρ, then G is $(\rho+1)$-colorable.

Proof

We will prove by induction on the number of vertices of G. Suppose that G be a graph with n vertices, then if we delete any vertex v (and the edges incident to it). Then, G remains with $n-1$ vertices whose largest vertex-degree is at most ρ. By our induction hypothesis, this graph is $(\rho+1)$-colorable; a $(\rho+1)$-coloring for G is, then, obtained by coloring v with a different color from the (at most ρ) vertices adjacent to v.

Theorem

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then, $v - e + r = 2$.

Proof

We proceed by induction on the edge e of a connected plane graph. There is only one connected graph of '0' edges, namely K_1. In this case, $v = 1, e = 0,$ and $r = 1$. Since $v - e + r = 2$, the base case of the induction holds.

Assume for a positive integer v that if H is a connected plane graph of v' vertices and e' edge, where $v' < v$ such that there are r' regions, then $v' - e' + r' = 2$. Let G be a connected plane graph of v vertices and e edges with r regions. We consider two cases.

Case 1. G is a tree. In this case, $e = v-1$ and $r = 1$. Thus, $v-e+r = v-(v-1)+1 = 2$, producting the desired result.

Case 2. G is not a tree. Since G is connected and is not a tree, G contains an edge e that is not a bridge. In G, the edge e is on the boundaries of two regions. So in $G-e$ these two regions merge into a single region. Since $G-e$ has v vertices, $e-1$ edges, and $r-1$ regions and $e-1<e$, it follows by the induction hypothesis that $v-(e-1)+(r-1) = 2$ and so $v-e+r = 2$.

Theorem

If G is a planar graph with vertices $v \geq 3$ and e edges, then $e \leq 3v-6$.

Proof

Since every graph with vertices $v \geq 3$, the inequality holds for $v = 3$. Assume that $v \geq 4$. Furthermore, assume that the planar graphs under consideration are connected, for otherwise edges can be added to produced a connected graph. Suppose that G is a connected planar graph with $v \geq 4$ and e edges and that there is a given planar embedding of G, resulting in r regions. By Theorem 3.3, $v-e+r = 2$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and suppose that we denote the number of edges on the boundary of $R_i (1 \leq i \leq r)$ by e_i. Then $e_i \geq 3$. Since each edge of G is on the boundary of at most two regions of G, it follows that

$$3r \leq \sum_{i=1}^{r} e_i \leq 2e.$$

Hence, $6 = 3v-3e+3r \leq 3v-3e+2e = 3v-e$ and $e \leq 3v-6$.

Lemma

If G is a planar graph, then G contains a vertex whose degree is at most 5.

Proof

Suppose that the degree of every vertex of G is at least 6. then the sum of the degrees of vertices would be at least 6v, where v is the number of vertices in G. Since the sum of all degrees in G is 2r, where e is the number of edges of G; thus $2e \geq 6v$ (or) $e \geq 3v$. But this contradicts the theorem, which states that $e \leq 3v-6$ for any planar graph with at least two edges. Thus, G must contain a vertex of degree at most 5.

Theorem

Every planar graph is 6-colorable.

Proof

We prove the theorem by induction on the number of vertices, the result being trivial for planar graphs with fewer than seven vertices. Suppose then that G is a planar graph with n vertices, and that all planar graphs with $n-1$ vertices are 6-colorable. Without loss of generality G can be assumed to be a simple graph, and so, by lemma 3.5, contains a vertex v whose degree is at most five, if we delete v, then the graph which remains has $n-1$ vertices

and it has 6-colorable. A 6-coloring for G is the obtained by coloring v with a different color from the (at most five) vertices adjacent to v.

Example

An exam schedule needs to be set up for the following courses: Calculus, Data structures, Discrete mathematics, European history, French, Physics, Psychology, and Shakespeare. The following pairs of courses (and only these) have students in common: Calculus and French (that is, there is at least one student who is taking both Calculus and French), Calculus and Psychology, Data structures and European history, Discrete mathematics and Physics, Discrete mathematics and Shakespeare, European history and French, European history and Shakespeare, French and Psychology, and Physics and Psychology. The exams must be scheduled in such a way that no student is required to take two exams on the same day. The problem is to determine the minimum number of examination days necessary, and to schedule the examinations.

This schedule problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for an exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.

Let Ca = Calculus,
Da = Data structures,
Dis = Discrete mathematics,
Euro = European history,
Fre = French,
Phy = Physics,
Psy = Psychology,
Sha = Shakespeare.

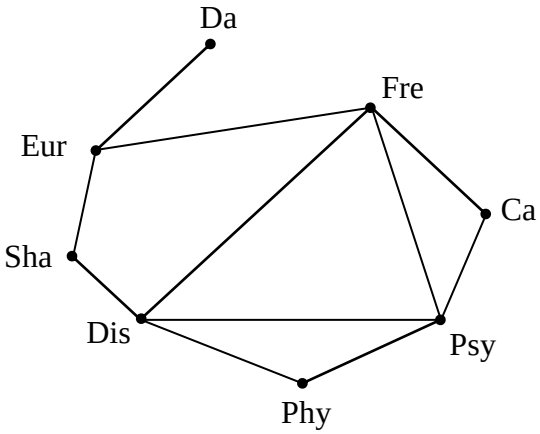

Figure 5 The graph representing the scheduling of exams

Since the chromatic number of this graph is 3, three time slots are needed. A coloring of the graph using three colors and the associated schedule are shown in Figure 6.

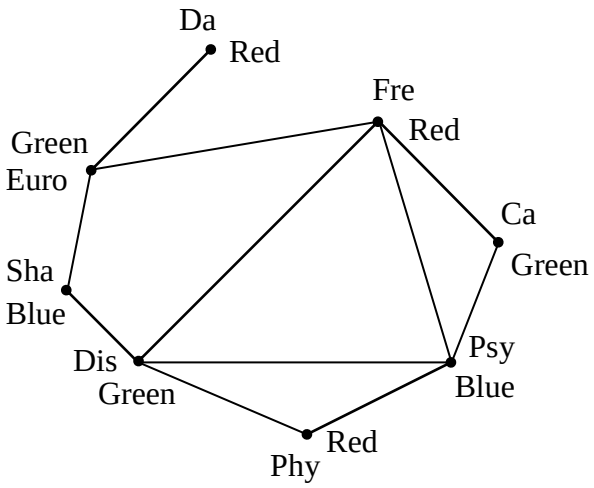

Figure 6 Using a coloring to schedule for exams

Time Period	Courses
I	Fre, Phy, Da
II	Ca, Dis, Euro
III	Sha, Psy

Acknowledgements

I am greatly thankful to Dr Aung Kyaw Thin, Rector and Dr Aye Aye Han, Pro-Rector of Banmaw University for the permission to the journal. I would like to express my thanks to Dr Khin San Aye, Professor and Head, Department of Mathematics, Banmaw University for her kind encouragement to submit this paper. I would like to thank Professor Dr Hnin Oo Lwin, Department of Mathematics, Banmaw University for her helpful advice to do this paper.

References

Bondy, J. A. and Murty, U. S. R., "Graph Theory with Applications", Macmillan Press Ltd, New York, (1976).

Grossman, J. W., "Discrete Mathematics", Macmillan Press Ltd, New York, (1990).

Rosen, K. H., "Discrete Mathematics and Its Applications", Mc Graw-Hill, New York, (2003).

Wilson, R. J., " Introduction to graph Theory", Longman Graph Limited, London, (1979).

Application of Number Theory in Cryptography

Kyaw San Lin*

Abstract

In this paper, some basic definitions, notations and concepts of number theory are expressed. Then, congruent modulo relation, equivalence class and the set of equivalence classes Z_n for integer n are defined and the relation between existence of multiplicative inverse of element in Z_n and being relatively prime with n is discussed. Moreover some definitions and terminologies concerned with cryptology are stated. Finally encryption and decryption of messages are illustrated with examples.

Introduction

Cryptography was concerned initially with providing secrecy for written messages. Cryptography depends on number theory and abstract algebra. In this paper, we shall introduce some basic concepts and techniques of cryptography. Messages can be encrypted and decrypted by using privative key as well as public key. But we shall state only privative key cryptography in this paper.

Some Results from Number Theory

Definitions

A nonzero integer a is said to **divide** an integer b if $b = ac$ for some integer c and we express it as a | b.

The following results can be obtained (i) a | b, b | c, then a | c,

(ii) a | b, b | c, then $a \mid b + c$, (iii) a | 0, a | a.

An integer d >0 is called **greatest common divisor** (gcd) of two nonzero integers a, b if (i) d | a, d | b and (ii) if c | a, c | b then c | d. We write $d = \gcd(a, b)$. If $\gcd(a, b) = 1$, then a and b are said to be **relatively prime** or **co-prime**. An integer p >1 is called a **prime number** if 1 and p are the only divisors of p. Let a, b, n, (n >0) be integers. We say that a is congruent to b modulo n if n divides $a - b$ and write $a \equiv b (\mathrm{mod}\, n)$. An **equivalence relation** on a set X is a relation $R \subset X \times X$ such that

(i) $(x, x) \in R$ for all $x \in X$ (**reflexive property**):

(ii) $(x, y) \in R$ implies $(y, x) \in R$ (**symmetric property**):

(iii) $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (**transitive property**).

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Given an equivalence relation R on a set X, we usually write $x \sim y$ instead of $(x, y) \in R$. A **partition** P of a set X is a collection of nonempty sets $X_1, X_2, ...$ such that $X_i \cap X_j = \varnothing$ for $i \neq j$ and $\bigcup_k X_k = X$. Let $\sim$ be an equivalence relation on a set X and let $x \in X$.

Then $[x] = \{y \in X : y \sim x\}$ is called the **equivalence class** of x.The integers mod n also partition Z into n different equivalence classes: we will denote the set of these equivalence classes by Z_n.

Theorem

Let Z_n be the set of equivalence classes of the integers mod n and $a, b \in Z_n$. Let a be a nonzero integer. Then, $\gcd(a, n) = 1$ if and only if there exists a multiplicative inverse b for a(mod n); that is, a nonzero integer b such that $ab \equiv 1 (\mathrm{mod}\, n)$.

Proof: See [2].

Introduction to Cryptography

Cryptography is the study of sending and receiving secret messages. The aim of cryptography is to send messages across a channel so only the intended recipient of the message can read it. The message to be sent is called the **plaintext** (M) message. The disguised message is called the **ciphertext** (C) . A **cryptosystem**, or **cipher**, has two parts: **encryption**, the process of transforming a plaintext message to a ciphertext message, and **decryption**, the reverse transformation of changing a ciphertext message into a plaintext message.

There are many different families of cryptosystems, each distinguished by a particular encryption algorithm. Cryptosystems in a specified cryptographic family are distinguished from one another by a parameter to the encryption function called a **key**. A classical cryptosystem has a single key, which must be kept secret, known only to the sender and the receiver of the message.

Privative Key Cryptography

In **single** or **privative key cryptosystems** the same key is used for both encrypting and decrypting messages. To encrypt a plaintext message, we apply to the message some function which is kept secret, say f. This function will yield an encrypted message. Given the encrypted form of the message, we can recover the original message by applying the inverse transformation f^{-1}.

(I) Monographic (Character) Ciphers

Cryptosystems are based on transforming each letter of plaintext into a different letter to produce the ciphertext. Such ciphers called **character**, **substitution** or **monographic ciphers**, since each letter is shifted individually to another letter by a substitution. First of all, let us define the numerical equivalents, as in Table 1, of the 26 English capital letters, since our operations will be on the numerical equivalents of letters, rather than the letters themselves.

Table 1.

A	B	C	D	E	F	G	H	I	J	K	L	M
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
0	1	2	3	4	5	6	7	8	9	10	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
13	14	15	16	17	18	19	20	21	22	23	24	25

Numerical equivalents of English capital letters

(i) Caesar cipher: A simple Caesar cipher uses the following substitution transformations: $f_3 \equiv m+3 (\mathrm{mod}\, 26),\ 0 \le m \le 25,$ and $f_3^{-1} \equiv c-3 (\mathrm{mod}\, 26),\ 0 \le c \le 26$ where $m \in M$ and $c \in C,$ and 3 is the key for both encryption and decryption.

(ii) Shift transformations: Slightly more general transformations are the following so called shift transformations:
$f_k \equiv m+k (\mathrm{mod}\, 26),\ 0 \le m,k \le 25,\ f_k^{-1} \equiv c-k (\mathrm{mod}\, 26),\ 0 \le c,k \le 26.$

(iii) Affine transformations: More general transformations are the following so called affine transformations: $f_{(a,b)} \equiv am+b (\mathrm{mod}\, 26),$ where the key $a, b \in Z,\ 0 \le a, b, m \le 26$ and $\gcd(a,26) = 1,$ together with $f_{(a,b)}^{-1} \equiv a^{-1}(c-b) (\mathrm{mod}\, 26),$ where a^{-1} is the multiplicative inverse of a modulo 26.

Example

By using the following affine transformations $f_{(7,21)} \equiv 7m+21 (\mathrm{mod}\, 26)$ and $f_{(7,21)}^{-1} \equiv 7^{-1}(c-21) (\mathrm{mod}\, 26),$ we can encrypt the plaintext message SECURITY and decrypt the ciphertext message VLXIJH as follows.

To encrypt the message SECURITY, we have

$S = 18,\ (7 \bullet 18+21) \mathrm{mod} 26 = 17 \ \ S \Rightarrow R,$

$E = 4,\ (7 \bullet 4+21) \mathrm{mod} 26 = 23 \ \ E \Rightarrow X,$

$C = 2,\ (7 \bullet 2+21) \mathrm{mod} 26 = 9 \ \ C \Rightarrow J,$

$U = 20,\ (7 \bullet 20+21) \mathrm{mod} 26 = 5 \ \ U \Rightarrow F,$

$R = 17,\ (7 \bullet 17+21) \mathrm{mod} 26 = 10 \ \ R \Rightarrow K,$

$I = 8,\ (7 \bullet 8+21) \mathrm{mod} 26 = 25 \ \ I \Rightarrow Z,$

$T = 19,\ (7 \bullet 19+21) \mathrm{mod} 26 = 24 \ \ T \Rightarrow Y,$

$Y = 24,\ (7 \bullet 24+21) \mathrm{mod} 26 = 7 \ \ Y \Rightarrow H.$

So the ciphertext message of the message SECURITY is RXJFKZYH.

To decrypt the message VLXIJH, we have

$$V = 21,\ [7^{-1}(21-21)]\bmod 26 = 0 \ \ V \Rightarrow A,$$

$$L = 11,\ [7^{-1}(11-21)]\bmod 26 = 6 \ \ L \Rightarrow G,$$

$$X = 23,\ [7^{-1}(23-21)]\bmod 26 = 4 \ \ X \Rightarrow E,$$

$$I = 8,\ [7^{-1}(8-21)]\bmod 26 = 13 \ \ I \Rightarrow N,$$

$$J = 9,\ [7^{-1}(9-21)]\bmod 26 = 2 \ \ J \Rightarrow C,$$

$$H = 7,\ [7^{-1}(7-21)]\bmod 26 = 24 \ \ H \Rightarrow Y.$$

Therefore the plaintext message for the message VLXIJH is AGENCY.

(II) Polygraphic (Block) Ciphers

Monographic ciphers can be made more secure by splitting the plaintext into groups of letters (rather than a single letter), and then performing the encryption and decryption on these groups of letters. This block technique is called **block ciphering**. Block cipher is also called a **polygraphic cipher**. Block ciphers may be described as follows:

(i) Split the message M into blocks of n-letters (when $n = 2$ it is called a **digraphic cipher**) $M_1, M_2, ..., M_j$; each block M_i for $1 \le i \le j$ is a block consisting of n letters.

(ii) Translate the letters into their numerical equivalents and form the ciphertext:

$C_i \equiv AM_i + B(\bmod 26),\ i = 1, 2, ..., j$ where (A, B) is the key, A is an invertible $n \times n$ matric with $\gcd(\det(A), 26) = 1, B = (B_1, B_2, ..., B_n)^T, C = (c_1, c_2, ..., c_n)^T$ and $M_i = (m_1, m_2, ..., m_n)^T$. For simplicity, we just consider $C_i \equiv AM_i (\bmod 26)$.

(iii) For decryption, we perform $M_i \equiv A^{-1}(C_i - B)(\bmod 26)$, where A^{-1} is the inverse matric of A. Again, for simplicity, we just consider $M_i \equiv A^{-1}C_i (\bmod 26)$.

Example

Let M = YOUR PIN NO IS FOUR ONE TWO SIX be the plaintext and $n = 3$. Let also the encryption matrix be

$$A = \begin{pmatrix} 11 & 2 & 19 \\ 5 & 23 & 25 \\ 20 & 7 & 17 \end{pmatrix}.$$

Then, the encryption and decryption of the message can be described as follows.

(i) Split the message M into blocks of 3-letters and translate these letters into their numerical equivalents:

Y	O	U	R	P	I	N	N	O	I	S	F
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
24	14	20	17	15	8	13	13	14	8	18	5

O	U	R	O	N	E	T	W	O	S	I	X
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
14	20	17	14	13	4	19	22	14	18	8	23

(ii) Encrypt these eight blocks in the following way:

$$C_1 = A\begin{pmatrix}24\\14\\20\end{pmatrix} = \begin{pmatrix}22\\6\\8\end{pmatrix}, \quad C_2 = A\begin{pmatrix}17\\15\\8\end{pmatrix} = \begin{pmatrix}5\\6\\9\end{pmatrix}, \quad C_3 = A\begin{pmatrix}13\\13\\14\end{pmatrix} = \begin{pmatrix}19\\12\\17\end{pmatrix}, \quad C_4 = A\begin{pmatrix}8\\18\\5\end{pmatrix} = \begin{pmatrix}11\\7\\7\end{pmatrix},$$

$$C_5 = A\begin{pmatrix}14\\20\\17\end{pmatrix} = \begin{pmatrix}23\\19\\7\end{pmatrix}, \quad C_6 = A\begin{pmatrix}14\\13\\4\end{pmatrix} = \begin{pmatrix}22\\1\\23\end{pmatrix}, \quad C_7 = A\begin{pmatrix}19\\22\\14\end{pmatrix} = \begin{pmatrix}25\\15\\18\end{pmatrix}, \quad C_8 = A\begin{pmatrix}18\\8\\23\end{pmatrix} = \begin{pmatrix}1\\17\\1\end{pmatrix}.$$

(iii) Translating these into letters, we get the ciphertext C:

22	6	8	5	6	9	19	12	17	11	7	7
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
W	G	I	F	G	J	T	M	R	L	H	H

23	19	7	22	1	23	25	15	18	1	17	1
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
X	T	H	W	B	X	Z	P	S	B	R	B

Therefore, the ciphertext message (C) of the given plaintext message is WGIF GJT MR LH HXTH WBX ZPS BRB.

(iv) To recover the message M from C, we first compute A^{-1} modulo 26:

$$A^{-1} = \begin{pmatrix}11 & 2 & 19\\5 & 23 & 25\\20 & 7 & 17\end{pmatrix}^{-1} = \begin{pmatrix}10 & 23 & 7\\15 & 9 & 22\\5 & 9 & 21\end{pmatrix}$$

And then perform $M_i = A^{-1}C_i \pmod{26}$ as follows:

$$M_1 = A^{-1}\begin{pmatrix}22\\6\\8\end{pmatrix} = \begin{pmatrix}24\\14\\20\end{pmatrix}, \quad M_2 = A^{-1}\begin{pmatrix}5\\6\\9\end{pmatrix} = \begin{pmatrix}17\\15\\8\end{pmatrix}, \quad M_3 = A^{-1}\begin{pmatrix}19\\12\\17\end{pmatrix} = \begin{pmatrix}13\\13\\14\end{pmatrix}, \quad M_4 = A^{-1}\begin{pmatrix}11\\7\\7\end{pmatrix} = \begin{pmatrix}8\\18\\5\end{pmatrix},$$

$$M_5 = A^{-1}\begin{pmatrix}23\\19\\7\end{pmatrix} = \begin{pmatrix}14\\20\\17\end{pmatrix}, \quad M_6 = A^{-1}\begin{pmatrix}22\\1\\23\end{pmatrix} = \begin{pmatrix}14\\13\\4\end{pmatrix}, \quad M_7 = A^{-1}\begin{pmatrix}25\\15\\18\end{pmatrix} = \begin{pmatrix}19\\22\\14\end{pmatrix}, \quad M_8 = A^{-1}\begin{pmatrix}1\\17\\1\end{pmatrix} = \begin{pmatrix}18\\8\\23\end{pmatrix}.$$

(v) Translating these into letters, we get the plaintext M:

24	14	20	17	15	8	13	13	14	8	18	5
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
Y	O	U	R	P	I	N	N	O	I	S	F

14	20	17	14	13	4	19	22	14	18	8	23
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
O	U	R	O	N	E	T	W	O	S	I	X

So, the original plaintext message YOUR PIN NO IS FOUR ONE TWO SIX can be obtained.

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References

[1] Khanna, V. K. and Bhambri, S. K., (1998) "A Course in Abstract Algebra", Vikas Publishing House PVT Ltd., New Delhi.

[2] Thomas W. J., (2011) "Abstract Algebra Theory and Applications ", Free Software Foundation.

The Dual of an Operator Spaces

M. Roi Seng[1], Khon Mai[2], Kyaw Htet Lynn[3]

Abstract

In this paper, we introduce the operator spaces and present some notions of operator spaces. And then, we study the concept of a dual of an operator spaces. Finally, we discuss some fundamental properties of dual operator spaces.

Introduction

The dual space is an important concept in the study of functional analysis.And so, one of the interesting areas of operator spaces is duality. Now, in this paper we present a dual of operator space which is again an operator space and discuss the operator space versions of usual Banach duality of subspaces and quotients.

Operator Spaces

An operator space is simply a Banach space together with an isometric linear embedding into the space B(H) of all bounded operators on a Hilbert space H. Now, in this section, we present some basic notions and examples of operator spaces. In what follows unless otherwise stated, V denote a vector space over the real field $\mathbb{R}$ or complex field $\mathbb{C}$ and $\mathbb{N}$ denotes the set of all positive integers. Firstly, we recall some matrix notions which is important tools for operator space.

Definition

For positive integers $m,n, M_{m,n}(V)$ denotes the vector space of all m by n matrices with entries in V. Then $M_{m,n}(V)$ is called a **matrix space** of a vector space V. In particular $M_m(V) = M_{n,n}(V)$ is called n^{th} level matrix space of V. When $V =,$ we simply write $M_{m,n}$ for $Mm,n(\mathbb{C})$ and M_n for $Mn(\mathbb{C})$. For $n = 1,$ we identify $M_{m,n}(V) = V.$

Definition

Let $v = \left[v_{ij}\right] \in M_{m,n}(V)$ and $v' = \left[v'_{kl}\right] \in M_{p,q}(V).$ Then the **direct sum** $v \oplus v'$ is defined by

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$$v \oplus v' = \begin{bmatrix} v_{ij} & 0 \\ 0 & v'_{kl} \end{bmatrix} \in M_{m+p,n+q}(V).$$

We also define

$$M_{m,n}(V) \oplus M_{p,q}(V) = \left\{ v \oplus v' : v \in M_{m,n}(V), v' \in M_{p,q}(V) \right\}.$$

Definition

Let V be an algebra and $v = \left[v_{ij} \right] \in M_{m,n}(V), v' = \left[v'_{kl} \right] \in M_{p,q}(V)$. Then the **Kronecker Product** $v \otimes v'$ is defined by

$$v \otimes v' = \left[v_{ij} v' \right] = \left[v_{ij} v'_{kl} \right] \in M_{mp,nq}(V).$$

We also define

$$M_{m,n}(V) \otimes M_{p,q}(V) = \left\{ v \otimes v' : v \in M_{m,n}(V), v' \in M_{p,q}(V) \right\}.$$

Definition

Let V be a vector space. A **matrix norm** on V is a family of norm $\|\cdot\|_n : M_n(V) \to \mathbb{R}$, one on each matrix level $M_n(V) = M_n \otimes V$ for $n \in \mathbb{N}$ which satisfies:

$(R_1) \|\alpha \; v \; \beta\| \leq \|\alpha\| \|v\| \|\beta\|,$

$(R_2) \|v \oplus w\| \leq \max \left\{ \|v\|, \|w\| \right\}$, for all $v \in M_n(V), \alpha \in M_{m,n}, \beta \in M_{n,m}$ and $w \in M_m(V)$.

Definition

Let V be a vector space. For each $n \in \mathbb{N}, M_n(V)$ together with the matrix norm is called an **operator space**.

Examples

(i) The space of all bounded linear operator on a Hilbert space H, B(H), is an operator space.

(ii) Any C*-algebra A is an operator space.

Definition

Given operator spaces V and W and a linear map $\varphi : V \to W$, there are corresponding linear maps $\varphi_n : M_n(V) \to M_n(W)$ defined by

$$\varphi_n([v_{ij}]) = \left[\varphi(v_{ij}) \right], \text{ for all } \left[v_{ij} \right] \in M_n(V).$$

The **completely bounded norm** of φ is defined by $\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$, which may be infinite.

φ is called a **completely bounded map** if $\|\varphi\|_{cb} < \infty$.

If Y is another operator space and $\psi : W \to Y$ is completely bounded, then the composition $\psi \cdot \varphi : V \to Y$

is completely bounded and we have

$$\|\psi \cdot \varphi\|_{cb} \le \|\varphi\|_{cb} \|\psi\|_{cb}.$$

Definition

Let V and W be operator spaces and $\varphi : V \to W$ be a linear map. φ is called a **complete contraction** if $\|\varphi\|_{cb} \le 1$. We define φ to be a **complete isometry** if each mapping $\varphi_n : M_n(V) \to M_n(W)$ is an isometry. φ is called a **complete quotient mapping** if each φ_n is a quotient mapping, that is, for each $y \in M_n(W)$,

$$\|y\| = \inf\{\|x\| : x \in \varphi_n^{-1}(y)\} \text{ for each } n \in \mathbb{N}.$$

We define φ to be a **complete isomorphism** if it is a linear isomorphism and $\|\varphi\|_{cb}$, $\|\varphi^{-1}\|_{cb} < \infty$. We say that the operator spaces V and W are **completely isometric** (**completely isomorphic**) if there is a complete isometry (complete isomorphism) from V onto W.

The following is Ruan's Theorem whose proof can be found in [1].

Theorem(Ruan)

Suppose that V is a vector space and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(V)$. Then V is completely isometrically isomorphic to a linear subspace of B(H) if and only if these norms satisfy conditions (R_1) and (R_2).

In [1] Pisier introduced the operator space version of the Banach-Mazur distance, which is defined as follows:

Definition

If V and W are operator spaces and V is completely isomorphic to W, then we define the Banach-Mazur distance

$$d_{cb}(V, W) = \inf\{\|t\|_{cb}, \|t^{-1}\|_{cb}\},$$

where the infimum is taken over all completely bounded isomorphisms $t : V \to W$, and define $d_{cb}(V, W) = \infty$ if V is not completely isomorphic to W.

We note that if V and W are completely isometric, then $d_{cb}(V,W)=1$.

Definition

Let V be an operator space and W be a closed subspace of V. Then the inclusion $M_n(W)\subseteq M_n(V)$ and the corresponding relative norms on V determine an operator space matrix norm on W.

By using the identification

$M_n(V/W)\subseteq M_n(V)/M_n(W)$ to define a norm on $M_n(V/W)$.

If $q:V\to V/W$ is the quotient mapping, then for each $n\in\mathbb{N}$ we define

$q_n : M_n(V)\to M_n(V/W)$ with norm

$\|\tilde{v}\|=\inf\{\|v\| : v\in M_n(V),(V)=\tilde{v}\}$, for all $\tilde{v}\in M_n(V/W)$.

Proposition

If W is a closed subspace of an operator space V, then V/W is an operator space.

Proof

Given $\alpha\in M_{n,m}$, $\beta\in M_{m,n}$ and $\tilde{v}\in M_m(V/W)$, there exists a $v\in M_m(V)$ such that

$q_m(v)=\tilde{v}$ and $\|v\|<\|\tilde{v}\|+\varepsilon$.

It follows that $q_n(\alpha\, v\,\beta)=\alpha\,\tilde{v}\,\beta$, and thus

$$\begin{aligned}\|\alpha\,\tilde{v}\,\beta\| &\le \|\alpha\, v\,\beta\|\\ &\le \|\alpha\|\|v\|\|\beta\|\\ &\le \|\alpha\|(\|\tilde{v}\|+\varepsilon)\|\beta\|.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we obtain (R_1).

On the other hand, given $\tilde{w}\in M_m(V/W)$ and an element $w\in M_n(V)$ with $q_n(w)=\tilde{w}$ and $\|w\|<\|\tilde{w}\|+\varepsilon$, it follows that $q_{m+n}(v\oplus w)=\tilde{v}\oplus\tilde{w}$, and thus

$$\begin{aligned}\|\tilde{v}\oplus\tilde{w}\| &\le \|v\oplus w\|=\max\{\|v\|,\|w\|\}\\ &\le \max\{\|\tilde{v}\|,\|\tilde{w}\|\}+\varepsilon.\end{aligned}$$

Again since $\varepsilon>0$ is arbitrary, we obtain (R_2).

By Ruan Theorem, V/W is an operator space.

Duality of Operator Spaces

In this section, we discuss the duality of operator space. Firstly, we present the mapping space which is important concept for study of operator space dual.

Definition

Let V and W be operator spaces. Then the space of all completely bounded maps from V into W is denoted by $CB(V,W)$.

Each matrix $\varphi = \left[\varphi_{ij}\right] \in M_n(CB(V,W))$

determines a mapping

$$\varphi : V \to M_n(W)$$

by letting $\varphi(v) = \left[\varphi_{ij}(v)\right]$, for $v \in V$.

Proposition

Let V and W be operator spaces. Then the space $CB(V,W)$ is an operator space.

Proof

We use the linear identification

$M_n(CB(V,W) \cong CB(V,M_n(W))$, and the completely bounded norm on the second space to define a norm on $M_n(CB(V,W))$.

Then the equation

$M_n(CB(V,W) = CB(V,M_n(W))$ holds completely isometrically.

By Ruan Theorem, $CB(V,W)$ is an operator space.

Definition

Let V be an operator space. The **operator space dual** of V is defined as $V^* = CB(V,\mathbb{C})$.

Each matrix $f = \left[f_{ij}\right] \in M_n(V^*)$ determines a linear mapping $f : V \to M_n$, where $f(v) = \left[f_{ij}(v)\right]$.

This gives us a linear isomorphism

$$M_n(V^*) \cong CB(V,M_n).$$

The completely bounded norm on the second space defines a norm on $M_n(V^*)$.

Thus we have the isometric identification

$$M_n(V^*) = CB(V, M_n).$$

For given $f \in M_n(V^*)$, we have

$$\|f\| = \sup\{\|f_n(v)\| : v \in M_n(V),\ \|v\| \le 1\} \tag{1}$$

and for $v \in M_n(V)$,

$$\|v\| = \sup\{\|f_n(v)\| : f \in CB(V, M_n),\ \|f\|_{cb} \le 1\}. \tag{2}$$

Definition

An operator space W is said to be a **dual operator space** if W is completely isometrically isomorphic to the operator space dual V^* of an operator space V.

Proposition

If V is an operator space, then its dual space V^* is an operator space.

Proof

We suppose that

$$f \in M_m(V^*),\ \alpha \in M_{n,m} \text{ and } \beta \in M_{m,n}.$$

Then
$$\begin{aligned}\|(\alpha f \beta)_r\| &= \|(\alpha \otimes I_r) f_r (\beta \otimes I_r)\| \\ &\le \|\alpha \otimes I_r\| \|f_r\| \|\beta \otimes I_r\| \\ &\le \|\alpha\| \|f\|_{cb} \|\beta\|\end{aligned}$$

and hence $\|\alpha f \beta\|_{cb} \le \|\alpha\| \|f_\gamma\|_{cb} \|\beta\|.$

Then we have (R_1).

On the other hand, given

$$f \in M_m(V^*),\ g \in M_n(V^*) \text{ and } v \in M_r(V) \text{ with } \|v\| \le 1.$$

$$\begin{aligned}\|(f \oplus g)_r(v)\| &= \|[f(v_{ij}) \oplus g(v_{ij})]\| \\ &= \|f_r(v) \oplus g_r(v)\| \\ &\le \max\{\|f_r(v)\|, \|g_r(v)\|\} \\ &\le \max\{\|f\|_{cb}, \|g\|_{cb}\}\end{aligned}$$

and hence $\|f \oplus g\|_{cb} \le \max\{\|f\|_{cb}, \|g\|_{cb}\}$.

Then we have (R_2).

By Ruans Theorem, V^* is an operator space.

Definition

Given an operator space V, we define the **canonical inclusion** $i_v : V \to V^{**}$ by

$$\langle i_v(v), f \rangle = \langle f, v \rangle,$$

where V^{**} is the dual operator space of V^*.

Proposition

For any operator space V, the canonical inclusion

$$i_v : V \to V^{**}$$

is completely isometric.

Proof

For any $v \in M_n(V)$ and $f \in M_n(V^*)$, $\left((i_v)_n(v)_n\right)(f) = \left[i_v(v_{ij})(f_{kl})\right] = \left[f_{kl}(v_{ij})\right]$.

It follows from equation (1) and (2) of the definition of operator space dual that

$$\begin{aligned}\|(i_v)_n(v)\| &= \sup\left\{\|f_{kl}(v_{ij})\| : v \in M_n(v),\ \|v\| \le 1\right\} \\ &= \sup\left\{\|(i_v)_n(v)_n(f)\| : f \in CB(V, M_n),\ \|f\|_{cb} \le 1\right\} \\ &= \|v\|.\end{aligned}$$

Thus $(i_v)_n$ is isometric for each n.

Therefore i_v is a complete isometry.

Proposition

Given operator spaces V and W, if $\varphi : V \to W$ is a completely bounded mapping, then the adjoint mapping φ^* is a completely bounded from W^* to V^* with $\|\varphi^*\|_n = \|\varphi_n\|$,

for all $n \in \mathbb{N}$ and $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$.

Proof

Given for any $v \in M_n(V)$ and $g \in M_m(W^*)$, we have

$$\begin{aligned}\|(\varphi^*)_n\| &= \sup\left\{\left\|\langle\langle(\varphi^*)_n(g), v\rangle\rangle\right\|\right\} \\ &= \sup\left\{\left\|\langle\langle g, \varphi_n(v)\rangle\rangle\right\|\right\} \\ &= \|\varphi_n\|,\end{aligned}$$

where the supremum is the taken over all $g \in M_n(V^*)$ and $v \in M_n(V)$ of norm ≤ 1.

Consequently, we have $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$.

From the above Proposition, we immediately have the following Corollary.

Corollary

Given operator spaces V and W, and a completely bounded mapping $\varphi : V \to W,$ then the completely bounded mapping $\varphi^* : W^* \to V^*$ is a complete isometry. That is,

$\phi^* : CB(V, W) \to CB(W^*, V^*)$ is a complete isometry.

Definition

Let V be operator space and W be a closed subspace of V. If V^* is the dual operator space, then $W^{\perp} = \{f \in V^* : f(v) = 0, \text{ for all } v \in W\}$ is called **annihilator** of W.

Proposition(Duality of subspaces and quotients)

If W is a closed subspace of an operator space V, then we have the complete isometries

$(V / W)^* = W^{\perp}$ and $W^* = V^* / W^{\perp},$

where $W^{\perp}$ is the annihilator of W.

Proof

The dual of the inclusion map $i : W \to V$ will be a complete quotient map $i^* : V^* \to W^*,$ which induces a complete isometry

$$W^* \cong V^* / Ker(i^*) = V^* / W^{\perp}.$$

Similarly, the dual of the canonical quotient map $q : V \to V / W$ is the canonical complete isometry $q^* : (V / W)^* \to V^*.$

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References

[1] Effros, E.G., & Ruan, Z. J., "Operator Spaces", Oxford University Press, Oxford (2000).

[2] Effros, E.G., & Ruan, Z. J., "On the abstract Characterization of Operator Spaces", Proc. Amer. Math. Soc. 199, 579-589 (1993).

[3] Blecher, D. P., "The Standard dual of an operator space", Pacific J. Math. 153-15-30, (1992).

Characterization of Atomic Decompositions, Banach Frames and X_d-frames in Banach Spaces

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Abstract

This paper is mainly concerned with the atomic decompositions, Banach frames and X_d- frames.

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1 Related concepts of atomic decompositions and Banach frames

Throughout this paper, X and Y denote Banach spaces, X^* denotes the dual of X, X_d denotes a Banach space of scalar valued sequences unless otherwise stated.

1.1 Definition:

A Banach space X_d is called ***BK-space*** if the coordinate functionals are continuous.

1.2 Definitions

Let X be a Banach space, X_d be a BK-space and let $(y_i)_{i=1}^{\infty}$ be the sequence of vectors of X^* and $(x_i)_{i=1}^{\infty}$ be the sequence of vectors of X. The ordered pair $\left((y_i)_{i=1}^{\infty},(x_i)_{i=1}^{\infty}\right)$ is called an ***atomic decomposition*** of X with respect to X_d if

(i) $\left(\langle x, y_i\rangle\right)_{i=1}^{\infty} \in X_d$, for each $x \in X$,

(ii) There exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \left\|\left(\langle x, y_i\rangle\right)_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X, \text{ for each } x \in X,$$

(iii) $x = \sum_{i=1}^{\infty}\langle x, y_i\rangle x_i$, for each $x \in X$.

The constants A and B are called ***atomic bounds*** for $\left((y_i)_{i=1}^{\infty},(x_i)_{i=1}^{\infty}\right)$.

1.3 Definitions

Let X be a Banach space, X_d be a BK-space and let $(y_i)_{i=1}^{\infty}$ be the sequence of vectors of X^*. The sequence $\left((y_i)_{i=1}^{\infty},S\right)$ is called a ***Banach frame*** for X with respect to X_d if

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(i) $\left(\langle x, y_i \rangle\right)_{i=1}^{\infty} \in X_d$, for each $x \in X$,

(ii) There exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \left\|\left(\langle x, y_i \rangle\right)_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X, \text{ for each } x \in X,$$

(iii) There exists bounded linear operator $S: X_d \to X$ so that

$$S\left(\langle x, y_i \rangle\right)_{i=1}^{\infty} = x, \text{ for each } x \in X.$$

It turns out that there is a natural relationship between atomic decomposition and Banach frame. Namely, a Banach frame is an atomic decomposition if and only if the unit vectors form a basis for the space X_d .

1.4 Definition

An ordered sequence $\left(x_i\right)_{i=1}^{\infty}$ in a Banach space X is called a ***Schauder basis*** for X if for each x in X there is a unique sequence $\left(\alpha_i\right)_{i=1}^{\infty}$ of scalars such that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i.$$

1.5 Proposition: Let X be a Banach space, X_d be a BK-space and let $\left(y_i\right)_{i=1}^{\infty}$ be the sequence of vectors from X^* and $S: X_d \to X$ be given. Let $\left(e_i\right)_{i=1}^{\infty}$ be the unit vectors in X_d . Then, the following conditions are equivalent:

(i) $\left(\left(y_i\right)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d and $\left(e_i\right)_{i=1}^{\infty}$ is a Schauder basis for X_d .

(ii) $\left(\left(y_i\right)_{i=1}^{\infty}, S\left(e_i\right)_{i=1}^{\infty}\right)$ is an atomic decomposition of X with respect to X_d .

Proof: $(i) \Rightarrow (ii)$: Assume (i) holds. We have

(i) $\left(\langle x, y_i \rangle\right)_{i=1}^{\infty} \in X_d$, for each $x \in X$.

(ii) There exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \left\|\left(\langle x, y_i \rangle\right)_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X, \text{ for each } x \in X.$$

(iii) There exists bounded linear operator $S: X_d \to X$ so that

$$S\left(\langle x, y_i \rangle\right)_{i=1}^{\infty} = x, \text{ for each } x \in X.$$

Moreover, $(e_i)_{i=1}^{\infty}$ is a Schauder basis for X_d , it follows that there exists a unique sequence $(\alpha_i)_{i=1}^{\infty}$ of scalars such that

$$(\alpha_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \alpha_i e_i ..$$

Let $x_i = Se_i \in X$, for each i.

For $\left((y_i)_{i=1}^{\infty}, S(e_i)_{i=1}^{\infty}\right)$ is an atomic decomposition of X with respect to X_d , it suffices to show

$$x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i \text{ , for each } x \in X.$$

It follows easily that $x = S(\langle x, y_i \rangle)$

$$= S(\langle x, y_i \rangle) \sum_{i=1}^{\infty} e_i$$

$$= \sum_{i=1}^{\infty} \langle x, y_i \rangle Se_i$$

$$= \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i \text{ ,for each } x \in X.$$

$(ii) \Rightarrow (i)$: Assume that (ii) holds. We have

(i) $(\langle x, y_i \rangle)_{i=1}^{\infty} \in X_d$, for each $x \in X$.

(ii) There exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \left\|(\langle x, y_i \rangle)_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X, \text{ for each } x \in X.$$

(iii) $x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i$, for each $x \in X$.

For $\left((y_i)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d , it suffices to prove $S: X_d \to X$ is bounded and linear.

Let $S: X_d \to X$ be given by $S(\langle x, y_i \rangle)_{i=1}^{\infty} = x,$. We shall clearly see that S is a bounded and linear.

For all scalars α, β and for all $x, z \in X$, we have

$$S\left(\alpha\langle x, y_i\rangle + \beta\langle z, y_i\rangle\right) = S\left(\langle \alpha x, y_i\rangle + \langle \beta z, y_i\rangle\right)$$

$$= S\left(\langle \alpha x + \beta z, y_i\rangle\right)$$

$$= \alpha x + \beta z$$

$$= \alpha S\left(\langle x, y_i\rangle\right) + \beta S\left(\langle z, y_i\rangle\right)$$

and $\left\|S\left(\langle x, y_i\rangle\right)\right\|_X = \|x\|_X = \left\|\left(\langle x, y_i\rangle\right)\right\|_{X_d}$, for each $x \in X$.

Now $\left(\left(y_i\right)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d with frame bounds A= B= 1.

We have only to prove that $\left(e_i\right)_{i=1}^{\infty}$ is a Schauder basis for X_d. It is easily seen that

$$S\left(\langle x, y_i\rangle\right)_{i=1}^{\infty} = x = \sum_{i=1}^{\infty}\langle x, y_i\rangle x_i, \text{ for each } x \in X.$$

$$= \sum_{i=1}^{\infty}\langle x, y_i\rangle Se_i$$

$$= \sum_{i=1}^{\infty} S\langle x, y_i\rangle e_i. \qquad \square$$

Banach frames are quite general. In fact, every Banach space has a Banach frame defined on it as the next result shows.

1.6 Proposition: Every separable Banach space has a Banach frame with frame bounds A= B= 1.

Proof: Let X be a separable Banach space. By Hahn- Banach Theorem, we can choose a sequence $\left(y_i\right)_{i=1}^{\infty} \in X^*$ with $\|y_i\| = 1$. And so that for every $x \in X.$, we have

$\|x\| = \sup|y_i(x)|$.

That is, $\|x\| = \left\|\langle x, y_i\rangle\right\|_{X_d}$, for each $x \in X$.

Let X_d be a subspace of ℓ^{∞} given by

$$X_d = \left\{\left(\langle x, y_i\rangle\right) : x \in X\right\}.$$

That is, $\left(\langle x, y_i\rangle\right) \in X_d$, for each $x \in X$.

Let $S : X_d \to X$ be given by

$$S\left(\langle x, y_i\rangle\right)_{i=1}^{\infty} = x, \text{ for each } x \in X.$$

We see that S is linear and bounded as in the proof of Proposition 1.5.

Therefore , $\left(\left(y_i\right)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d with frame bounds A= B= 1. □

2 Related concepts of Banach frames and X_d-frames

Generalizations of frames to Banach spaces are the so called Banach frames and X_d- frames.

2.1 Definitions

Let X be a Banach space, X_d be a BK-space and let $\left(y_i\right)_{i=1}^{\infty}$ be the sequence of vectors of X^*. The sequence $\left(y_i\right)_{i=1}^{\infty}$ is called a ***X_d-frame*** for X if

(i) $\left(\langle x, y_i\rangle\right)_{i=1}^{\infty} \in X_d$, for each $x \in X$,

(ii) There exist two constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \left\|\left(\langle x, y_i\rangle\right)_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X, \text{ for each } x \in X,$$

The constants A and B are called ***the lower and upper X_d-frame bounds*** respectively.

If at least (i) and the upper condition in inequality from (ii) are satisfied , the sequence $\left(y_i\right)_{i=1}^{\infty}$ is called an ***X_d - Bessel sequence*** for X with bound B.

We first characterize the Banach space X which has an X_d-frame with respect to a given BK-space X_d.

2.2 Theorem

Let X be a Banach space and X_d be a BK-space. Then, there exists an X_d-frame for X if and only if X is isomorphic to a subspace of X_d.

Proof: From the definition of X_d-frame, if $\left(y_i\right)_{i=1}^{\infty}$ is an X_d-frame for a Banach space X, then the mapping $U : X \to X_d$ given by $U(x) = \left(\langle x, y_i\rangle\right)_{i=1}^{\infty}$, for each $x \in X$ is

an isomorphism of X into X_d .

Conversely , let X be a subspace of X_d and $(x_i)_{i=1}^{\infty}$ be the coordinate functionals (which are assumed to be continuous). Let $y_i = x_i|_X$.

Then for each $x \in X$,

(i) $\left(\langle x, y_i \rangle\right)_{i=1}^{\infty} = x \in X_d$ and

(ii) $\|x\|_X = \left\|\left(\langle x, y_i \rangle\right)_{i=1}^{\infty}\right\|_{X_d}$. □

Given an X_d-frame $(y_i)_{i=1}^{\infty}$, where X_d is a BK- space for which the canonical unit vectors form a basis, the next result clarifies which extra condition we need in order to ensure that $(y_i)_{i=1}^{\infty}$ is a Banach frame.

2.3 Definitions

A closed linear subspace Y of a Banach space X is said to be ***complemented subspace*** of X if there is a bounded linear projection from X onto Y.

In other words, Y is said to be a ***complemented subspace*** of X if there exists a closed subspace Z of X so that X is a direct sum of Y and Z.

2.4 Proposition

Suppose that X_d is a BK-space, $(y_i)_{i=1}^{\infty}$ be the sequence of vectors from X^* and X_d-frame for X. If the canonical unit vectors $(e_i)_{i=1}^{\infty}$ form a basis for X_d, then the following conditions are equivalent:

(i) Range R(U) of the operator U is complemented in X_d .

(ii) There exists bounded linear operator S such that $\left((y_i)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d .

(iii) There exists an X_d^* - Bessel sequence $(x_i)_{i=1}^{\infty} \subset X \subseteq X^{**}$ for X^* such that

$$x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i \quad \text{, for each } x \in X.$$

A reformulation of Proposition 2.4 gives a characterization of spaces X possessing Banach frames.

2.5 Theorem

A Banach space X has a Banach frame with respect to a sequence space X_d if and only if X is isomorphic to a complemented supspace of X_d.

Proof: Suppose that X has a Banach frame with respect to X_d . Then, there exists a bounded linear operator S such that $\left(\left(y_i\right)_{i=1}^{\infty}, S\right)$ is a Banach frame for X with respect to X_d. From the definition of X_d-frame, if $\left(y_i\right)_{i=1}^{\infty}$ is an X_d-frame for a Banach space X, then the mapping $U: X \to X_d$ given by

$$U(x) = \left(\langle x, y_i \rangle\right)_{i=1}^{\infty} = (y_i(x))_{i=1}^{\infty}, \text{ for each } x \in X$$

is an isomorphism of X into X_d .

From this ,

$$U^{-1}: R(U) \to X$$

is continuous on R(U). Then, U^{-1} can be extended to a bounded linear operator

$$V: X_d \to X.$$

Consider the bounded operator

$$P: X_d \to R(U)$$

defined by P = UV.

We get $P^2 = P$ by using

$$VU = I \text{ (on X)}.$$

For every $x \in X$,

$$Ux = UVUx = P(Ux) \in R(P).$$

Hence, R(U) = R(P).

That is, the range of U equals to the range of bounded projection. Thus, R(U) is complemented.

Conversely, suppose that X is isomorphic to a complemented subspace of X_d. Then, we will show how a Banach space can be constructed.

Let $T: X \to X_d$ be an isomorphism and let $P: X_d \to R(T)$ be a projection of X_d onto R(T). Define $S: X_d \to X$ by $Sx = T^{-1}Px$. Let $\left(e_i\right)_{i=1}^{\infty}$ be the coordinate functionals of X_d and $y_i = T^* e_i$.

For each $x \in X$,

$$y_i(x) = T^* e_i(x) = e_i(T x).$$

Hence, $T(x) = (y_i(x))$.

Since T is an isomorphism, it follows that $\left((y_i)_{i=1}^{\infty}, S\right)$ is a Banach frame for X

with respect to X_d .

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References

[1] P.G. Casazza : The art of frame theory, Taiwanese, J. Math. 4(2),(2000), 129-201.

[2] P.G. Casazza & D. Han & D. Larson : Frames for Banach spaces, Amer. Math. Soc. 247 (2000)149-182.

[3] J. Lindenstrauss & L. Tzafriri : Classical Banach Spaces I, Spinger-Verlag, Berlin, (1977).

[4] R. E. Megginson (1998). : An Introduction to Banach Space Theory, Springer-Verlag, New York,

[5] D.T. Stoeva : X_d-frames in Banach spaces and their duals, J. Math.F. A.(2008).

Some Applications of Eigenvalues and Eigenvectors

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Abstract

In this paper, firstly the basic concepts of matrices and determinants are introduced. And then, the important facts of the eigenvalues and eigenvectors are presented. Finally, some applications of eigenvalues and eigenvectors are described.

Introduction

The eigenvalues problems are the most important problems that link with matrix analysis. We shall see that eigenvalues and eigenvectors are associated with square matrix. Many applications involve the use of eigenvalues and vectors in the process of transforming a given matrix into a diagonal matrix. As the last, the calculation of the model population growth and finding a stable age distribution vector are expressed.

Learning Outcomes

On completion we should be able to obtain the eigenvalues and eigenvectors of 2×2 and 3×3 matrices. We should be able to diagonalize a matrix with distinct eigenvalues using the model matrix. We consider the model population growth by using an transition matrix and an age distribution vector. Also we state the quadratic forms to know the rotation of conic.

The Basic concepts of Matrices and Determinants

A **matrix** is simply a set of numbers arranged in a rectangle array. Then the array enclosed by round ( ) or square [ ] bracket.

$$A=\begin{pmatrix} 2 & 4 & -1 & 0 \\ 1 & 3 & 7 & 2 \end{pmatrix} \text{ is a } 2\times 4 \text{ matrix.}$$

It has 2 rows and 4 columns.

A matrix with the same numbers of rows and columns is called a **square matrix**.

For examples, $P=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $Q=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 2 & 3 & 3 \end{pmatrix}$ are the square matrix of order 2 and 3.

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A square matrix is a **diagonal matrix** if nondiagonal entries are all zeros. The main diagonal entries can be constants or zeros.

For examples, $C=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ and $D=\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$ are diagonals matrices.

A diagonal matrix whose diagonal entries are all equal is called a **scalar matrix**.

For examples, $A=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$ and $B=\begin{pmatrix}\frac{3}{2} & 0\\ 0 & \frac{3}{2}\end{pmatrix}$ are scalar matrices.

$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ are the unit matrix of dimension 2 and 3.

A square matrix possess an associated **determinant**. A determinant has a single value.

A 2×2 matrix $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ has an associated determinant $\det A=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$

$$=a_{11}a_{22}-a_{21}a_{12}.$$

A 3×3 matrix $B=\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$ has an associated determinant, $\det B=\begin{vmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{vmatrix}$.

This determinant can be calculated by "an expansion about the top row". $\det B=b_{11}\begin{vmatrix}b_{22} & b_{23}\\ b_{32} & b_{33}\end{vmatrix}-b_{12}\begin{vmatrix}b_{21} & b_{23}\\ b_{31} & b_{33}\end{vmatrix}+b_{13}\begin{vmatrix}b_{21} & b_{22}\\ b_{31} & b_{32}\end{vmatrix}$.

Eigenvalues and Eigenvectors

An **eigenvalue** of a square matrix A is a scalar λ such that $AX=\lambda X$ has a solution $X\neq 0$. The vector X is called an **eigenvector** of A corresponding to that eigenvalue λ.

To find the eigenvalue we use the characteristic equations of A, that is, $\det(\lambda I-A)=0$. Also we can find the corresponding eigenvectors by solving the equation $(\lambda I-A)X=0$ for the vector X, where I is a unit matrix the same dimensions as A.

Example

We can find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$.

From the equation $(\lambda I-A)X=0$ and $X=\begin{pmatrix}x\\ y\end{pmatrix}$,

we use the characteristic equation,

$$\det(\lambda I-A)=0,$$

$$\det\left\{\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}\right\}=0.$$

$$\begin{vmatrix}\lambda-1 & 0\\ -1 & \lambda-2\end{vmatrix}=0.$$

By expanding this determinant,

$$(\lambda-1)(\lambda-2)=0.$$

Hence the eigenvalues are $\lambda=1$ and $\lambda=2$.

So we have found two values of λ for this 2×2 matrix A.

Since the values are unequal, the eigenvectors are also distinct.

To each value of λ is the corresponding of an eigenvector. We now preceed to find the eigenvectors.

Case1:

If $\lambda=1,$ then our original eigenvalue problem becomes $AX=X.$

So $x=x$

$$x+2y=y.$$

$$x+y=0.$$

Then $x=-y\Rightarrow y=-x.$

Thus $X=\begin{pmatrix}x\\ -x\end{pmatrix}$ for any $x\neq 0.$

So the eigenvectors corresponding to eigenvalue $\lambda=1$ are proportional to $\begin{pmatrix}1\\ -1\end{pmatrix}$.

The normalized eigenvector of X is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$.

Case 2:

For the larger eigenvalue $\lambda = 2,$ our original eigenvalue problem becomes $AX = 2X$ which gives the following equations

$$x = 2x$$
$$x + 2y = 2y.$$

These equations imply that $x = 0$ and the variable y may take any value (except zero).

Thus the eigenvector corresponding to eigenvalue $\lambda = 2$ has the form $\begin{pmatrix} 0 \\ y \end{pmatrix}$.

The normalized eigenvector of X is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Therefore the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ has two eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2$ and

two associated normalized eigenvectors

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Some Applications of Eigenvalues and Eigenvectors

Population Growth

Matrices can be used to form model for population growth. The first step is to combine the population into age classes of equal duration.

Particularly, if the greatest life term of a number is L years, then the following n intervals perform the age classes.

$\left[0, \frac{L}{n}\right)$ *first age class*

$\left[\frac{L}{n}, \frac{2L}{n}\right)$ *second age class*

:

:

:

$\left[\frac{(n-1)L}{n}, L\right)$ n^{th} *age class*

The age distribution vector X represents the number of population members in each age class, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \begin{matrix} \textit{Number in } 1^{st} \textit{ class} \\ \textit{Number in } 2^{nd} \textit{ class} \\ \vdots \\ \textit{Number in } n^{th} \textit{ class} \end{matrix}$$

Over a period of $\frac{L}{n}$ years, the probability that a member of i^{th} age class will remain to become a member of the $(i+1)^{th}$ age class is given by p_i, where $0 \le p_i \le 1,\ i = 1,2,...,n-1.$

The average member of generation produced by a member of the i^{th} age class is given by b_i ,where $0 \le b_i,\ i = 1,2,...,n-1.$

These numbers can be written in matrix form

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & \dots & 0 & 0 \\ : & & & & & \\ : & & & & & \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}.$$

Multiplying this age transition matrix by the age distribution vector for a specific time period produces the age distribution vector for the next time period, that is, $Ax_i = x_{i+1}.$

Example

A population of rabbits has the following characteristics.

(i) Half of the rabbits survive their first year. Of those, half survive their second year.

The greatest life term is 3 years.

(ii) During the first year, the rabbits produce no offspring. The average number of offspring

is 6 during the second year and 8 during the third year.

The population now consists 24 rabbits in the first age class, 24 in the second and 20 in the third. We can compute the number of rabbits will be there in each age class in 1 year.

The current age distribution vector is

$$x_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \quad \begin{matrix} 0 \le age < 1 \\ 1 \le age < 2 \\ 2 \le age < 3 \end{matrix}$$

and the age transition matrix is $A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

After 1 year, the age distribution vector will be

$$x_2 = Ax_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}. \quad \begin{matrix} 0 \le age < 1 \\ 1 \le age < 2 \\ 2 \le age < 3 \end{matrix}$$

Example

To find a stable age distribution vector for the population in above example.

By using the characteristics equation is $|\lambda I - A| = 0$,

$$(\lambda + 1)^2(\lambda - 2) = 0.$$

Therefore $\lambda = -1$ or $\lambda = 2$.

Choosing the positive value $\lambda = 2$, then our original eigenvalue problem becomes $AX = 2X$.

Then
$$\begin{aligned} 6x_2 + 8x_3 &= 2x_1 \\ 0.5x_1 &= 2x_2 \\ 0.5x_2 &= 2x_3. \end{aligned}$$

The corresponding eigenvectors are of the form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}.$$

For instance, if $t = 2$, then the initial age distribution vector would be

$$X_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \quad \begin{matrix} 0 \le age < 1 \\ 1 \le age < 2 \\ 2 \le age < 3 \end{matrix}$$

and the age distribution vector for the next year would be

$$X_2 = AX_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix} \qquad \begin{matrix} 0 \le age < 1 \\ 1 \le age < 2 \\ 2 \le age < 3 \end{matrix}$$

The ratio of the three age classes is still 16:4:1 and so the percent of the population in each age class remain the same.

Quadratic Form

Eigenvalues and eigenvectors can be used to solve the rotation of axes problem. By classifying the graph of the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \qquad (1)$$

The graph is fairly straightforward as long as the equation has no xy-term,

that is, $b = 0$.

If the equation has an xy-term, then the classification is accomplished most easily by first performing a rotation of axes the eliminate the xy-term.

The resulting equation will be of the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0.$$

The coefficients a' and c' are the eigenvalues of the matrix $a = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$.

The expression $ax^2 + bxy + cy^2$ is called the **quadratic form** associated with the equation (1) and the matrix A is called the matrix of the quadratic from. Moreover, the matrix A will be **diagonal** if and only if its corresponding quadratic form has no xy-term.

Example

We find the matrix of a quadratic form associated with

(i) $4x^2 + 9y^2 - 36 = 0$, (ii) $13x^2 - 10xy + 13y^2 - 72 = 0$.

(i) Since $a = 4, b = 0$ and $c = 9$, the matrix is

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \qquad \text{Diagonal matrix (no xy-term)}$$

(ii) Since $a = 13, b = -10$ and $c = 13$, the matrix is

$$B = \begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix} \qquad \text{Nondiagonal matrix (xy-term).}$$

In standard form, the equation $4x^2+9y^2-36=0$ is

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

which is the equation of the ellipse shown in figure 1.

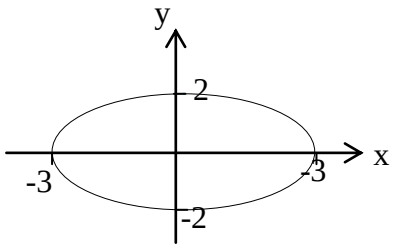

Figure 1

The graph of the equation $13x^2-10xy+13y^2-72=0.$ is similar. Infact when you rotate the x and y axes counterclockwise 45° to form a new $x'y'$-coordinate system, the equation forms $\frac{(x')^2}{3^2}+\frac{(y')^2}{2^2}=1$ which is the equation of the ellipse shown in figure 2.

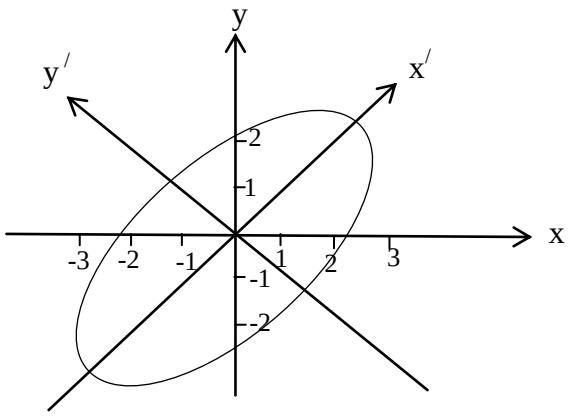

Figure 2

Principal Axes Theorem

For a conic whose is $ax^2+bxy+cy^2+dx+ey+f=0,$ the rotation given by $X=PX'$ eliminates the xy-term when P is an orthogonal matrix, $|p=1|$, with that

diagonalizes A. The matrix P is the form $P=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ where a gives the angles of rotation of the conic measured from $x-\text{axis}$ the positive to the positive $x'-\text{axis}$.

That is $P^{T}AP=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$, where λ_1 and λ_2 are eigenvalues of A.

The equation of the rotated conic is given by

$$\lambda_1(x')^2+\lambda_2(y')^2+\text{d}\quad\text{e}\ PX'+\text{f}=0.$$

Example

We perform a rotation of axes to eliminate the xy-term in the quadratic equation $13x^2-10xy+13y^2-72=0.$

The matrix of the quadratic form is $A=\begin{bmatrix}13 & -5\\ -5 & 13\end{bmatrix}.$

By using the characteristic equation is $|\lambda I-A|=0,$

$$(\lambda-8)(\lambda-18)=0.$$

The eigenvalues of A are $\lambda_1=8$ and $\lambda_2=18.$

So the equation of the rotated conic is $8(x')^2+18(y')^2-72=0.$

The standard form $\frac{(x')^2}{3^2}+\frac{(y')^2}{2^2}=1$ is the equation of an ellipse.

Example

We can perform a rotation to eliminate the xy-term in
$3x^2-10xy+3y^2+16\sqrt{2}\,x-32=0.$

The matrix of the quadratic form is $A=\begin{bmatrix}3 & -5\\ -5 & 3\end{bmatrix}.$

The eigenvalues of A are $\lambda_1=8$ and $\lambda_2=-2$ with corresponding eigenvectors of

$$x_1=(-1,1)\ \text{and}\ x_2=(-1,-1).$$

This implies that the matrix P is

$$P=\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}, \quad \text{where} \quad |P|=1.$$

The angle of rotation is $\theta=135^{\circ}$.

Then $[d \quad e]PX'=[16\sqrt{2} \quad 0]\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix}$

$$=-16x'-16y'.$$

The equation of the rotated conic is

$$8(x')^2-2(y')^2-16x'-16y'-32=0.$$

In standard form, the equation

$$\frac{(x'-1)^2}{1^2}-\frac{(y'+4)^2}{2^2}=1 \quad \text{is the equation of a hyperbolic.}$$

It graph is shown in figure 3.

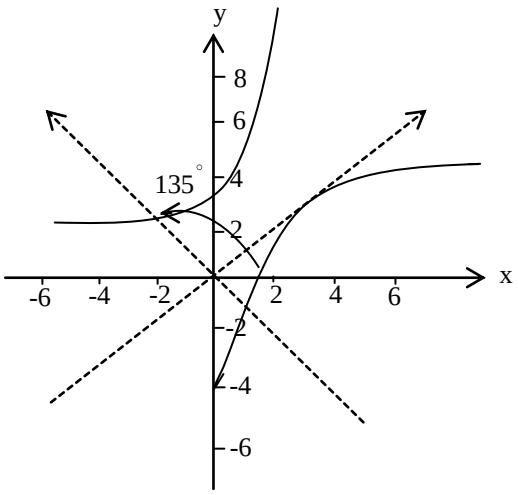

Figure 3

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References

[1] Kolman, B.,"Introductory Linear Algebra with applications", Macmillan.(1988).

[2] Kreyszing, E., " Introductory functional analysis with applications", John Willey & Sons. Inc. (1978).

[3] Lang, S., "Linear Algebra", Addison-Wesky, 1968.